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ON NON-ISENTROPIC STATIONARY SPATIAL AND PLANAR NON-STATIONARY DOUBLE-WAVES*

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A complete classification of planar non-stationary non-isentropic and spatial stationary gas flows of the double-wave type is given when there is a functional arbitrariness. Pressure and entropy are chosen as the independent variables in the hodograph space. It is shown that non-isentropic double-waves with a functional arbitrariness, which are not reduced to invariant solutions, only occur in the stationary spatial case in the case of gases with an equation of state $\tau = g(p) A(S)$. The flows which are described by these double-waves generalize Prandtl-Meyer waves to the spatial non-isentropic case. Only two forms of planar non-stationary non-isentropic gas flows of the double-wave type, which are not reduced to invariant solutions and have a functional arbitrariness, occur in the general solution of the Cauchy problem.

Double-waves have previously been classified within the framework of the equations of gas dynamics subject to additional assumptions regarding either potentiality /1/ or the weaker conditions of the rectilinearity of the contour lines of the flows being considered /2-4/. Papers concerned with this approach have been discussed in greater detail in /5/. An attempt has been made in /6/ to investigate double-waves without any additional assumptions. However, a complete investigation of the consistency of the resulting system of quasilinear first-order differential equations is rather difficult on account of the unwieldiness of the calculations. Particular solutions of this system were given in /6, 7/.

The overall investigation of travelling waves can be divided into two parts: with a constant and with a functional arbitrariness in the general solution of the Cauchy problem. In the case of solutions with a constant arbitrariness, the initial system of gas dynamic equations reduces to a completely integrable system as a result of two to three extensions. The conditions imposed on the independent variables in the space of the hodograph (the consistency conditions) are obtained for such systems by simple cross differentiation. The process by means of which these conditions are obtained is very unwieldy while the solution of the initial system has just a single constant arbitrariness in the general solution. The class of running waves have a functional arbitrariness in the general solution of the Cauchy problem is more important from the point of view of the solution of boundary value problems. Double-waves with a functional arbitrariness have been investigated in /8/ for the case of planar isentropic flows.

1. Double-waves of spatial steady-state gas flows. Stationary, non-isobaric and non-isentropic double-waves are considered which do not reduce to invariant solutions of the equations of gas dynamics

$$\frac{d\mathbf{u}}{dt} + \tau\nabla p = 0, \quad \frac{d\tau}{dt} - \tau \operatorname{div} \mathbf{u} = 0, \quad \frac{dS}{dt} = 0 \tag{1.1}$$

with an equation of state $\tau = \tau(p, S)$, $\tau_p \neq 0$, $\tau_S \neq 0$. Here $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity, p is the pressure, S is the entropy and τ is the specific volume. In the case of steady state flows $d/dt = u_\alpha \partial/\partial x_\alpha$ (here and subsequently, summation is carried out over a repeated Greek subscript).

If p and S are functionally independent, then, by selecting the functions p and λ as the independent parameters of the double-wave in the space of the hodograph (it is assumed that $p_{x_1}\lambda_{x_2} - p_{x_2}\lambda_{x_1} \neq 0$) without any loss in generality, it can be shown that the following assertions are valid. Either the double-waves (according to the reduction theorem /9/) are reduced to invariant solutions or, by carrying out the transformation $x_1 = P(p, \lambda, x_3)$, $x_2 = Q(p, \lambda, x_3)$ (as in /6/), we obtain a completely integrable system of first-order differential equations which has just a single constant arbitrariness in the solution. Hence, the pressure and the entropy are functionally independent in stationary non-isobaric non-isentropic double-waves which are not reduced to invariant solutions and have a functional arbitrariness.

Pressure and entropy were selected as the double-wave parameters. After introducing the new independent variable $\varphi = (\operatorname{div} u)/\tau_p$ from (1.1), we get

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$$\frac{dp}{dt} - \tau\varphi = 0, \quad \frac{dS}{dt} = 0, \quad u_{\alpha S}S_{x_\alpha} = H\varphi \tag{1.2}$$
$$(H \equiv \tau_p + u_{\alpha p}u_{\alpha p}), \quad \Phi_i \equiv p_{x_i} + u_{ip}\varphi = 0 \quad (i = 1, 2, 3)$$

Since the flow is non-isobaric $\varphi \neq 0, u_{\alpha p}u_{\alpha p} \neq 0$ (to be specific, it is assumed that $u_{1p} \neq 0$). It follows from (1.2) that

$$\tau + u_\alpha u_{\alpha p} = 0 \tag{1.3}$$

By the total differentiation of D_{x_i} with respect to the variable x_i and comparing the combinations, we get from Eq.(1.2) that

$$D_{x_i}\Phi_j - D_{x_j}\Phi_i = u_{jp}\varphi_{x_i} - u_{ip}\varphi_{x_j} + \varphi(u_{jpS}S_{x_i} - u_{ipS}S_{x_j}) -$$
$$\varphi^2(u_{jpp}u_{ip} - u_{ipp}u_{jp}) = 0 \tag{1.4}$$
$$D(H\varphi - u_{\alpha S}S_{x_\alpha})/Dt = H\,d\varphi/dt - \varphi(\tau u_{\alpha pS} + u_{\alpha p}u_{\beta p}u_{\beta S})S_{x_\alpha} +$$
$$\varphi^2(H^2 + \tau H_p) = 0 \tag{1.5}$$
$$D(\Phi_i)/Dt = u_{ip}d\varphi/dt + \tau\varphi_{x_i} + \varphi\zeta S_{x_i} -$$
$$\varphi^2(u_{ip}H - \tau u_{ipp}) = 0 \quad (i, j = 1, 2, 3; \quad i \neq j) \tag{1.6}$$
$$\zeta \equiv \tau_S + u_{\alpha p}u_{\alpha S}, \quad D/Dt = u_\alpha D_{x_\alpha}$$

Upon eliminating the derivatives $\varphi_{x_i}, \varphi_{x_j}$ from (1.4) with the aid of equations (1.6), we have

$$(\zeta u_{ip} - \tau u_{ipS})S_{x_j} - (\zeta u_{jp} - \tau u_{jpS})S_{x_i} = 0 \quad (i, j = 1, 2, 3; \quad i \neq j) \tag{1.7}$$

It is subsequently necessary to distinguish between the two cases $H \neq 0$ and $H = 0$.

1°. Let $H \neq 0$. If φ is expressed by rearranging the third equation of (1.2) and its value is substituted into the remaining equations of this system then, instead of (1.7), a homogeneous system of quasilinear equations is obtained and the equations

$$\tau u_{ipS} - \zeta u_{ip} = 0 \quad (i = 1, 2, 3) \tag{1.8}$$

follow from the fact that the reduction of the double-waves to invariant solutions is forbidden /9/. The existence of functions $F_i = F_i(p)$ such that

$$u_{ip} = F_i u_{1p} \quad (i = 2, 3) \tag{1.9}$$

follows from Eqs.(1.8).

Since $H \neq 0$, the existence of a determinant $u_iu_{jS} - u_ju_{iS}$ follows from the second and third equations of system (1.2). To be specific, it is assumed that $\Delta \equiv u_2u_{3S} - u_3u_{2S} \neq 0$.

The system of equations which the functions $\mathbf{v} = (p, S, \varphi)'$ must satisfy is written in the form of an overdetermined system of quasilinear differential equations

$$\mathbf{v}_{x_3} + G_3\mathbf{v}_{x_1} = \mathbf{f}_1, \quad \mathbf{v}_{x_2} + G_2\mathbf{v}_{x_1} = \mathbf{f}_2 \tag{1.10}$$
$$p_{x_1} + \varphi u_{1p} = 0, \quad S_{x_1}\varphi l_1 + \varphi_{x_1} = f_3$$
$$G_i = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & -\Delta_i & 0 \\ 0 & \varphi l_i & 0 \end{Vmatrix} \quad (i = 2, 3)$$

$$\Delta_2 = (u_3u_{1S} - u_1u_{3S})/\Delta, \quad \Delta_3 = (u_1u_{2S} - u_2u_{1S})/\Delta$$
$$l_i = u_{ip}\xi\lambda/(\tau H\Delta) + \zeta\Delta_i/\tau \quad (i = 2, 3)$$
$$l_1 = u_{1p}\xi\lambda/(\tau H\Delta) + \zeta/\tau$$
$$\lambda = \Delta(u_{1p} + u_{2p}\Delta_2 + u_{3p}\Delta_3), \quad \xi = \tau_S + 2u_{\alpha p}u_{\alpha S}$$

with functions $\mathbf{f}_1, \mathbf{f}_2$ and f_3 which are independent of the derivatives $\mathbf{v}_{x_i}(i = 1, 2, 3)$.

For the existence of a solution of system (1.10) which has a functional arbitrariness, it is necessary that

$$\xi\lambda(u_{ip} - u_{1p}\Delta_i) = 0 \quad (i = 2, 3) \tag{1.11}$$

Since $\tau H\varphi \neq 0$, the equation

$$\xi\lambda = 0 \tag{1.12}$$

follows from Eqs.(1.9), and (1.11).

When $\lambda = 0$, either a reduction to the two-dimensional case (possibly, by means of a rotation of the coordinate system) is obtained from (1.9) and the equation $u_{\alpha S}S_{x_\alpha} = H\varphi$ or the condition $H\varphi \neq 0$ is violated.

The case when $\xi = 0$, when some of the equations of system (1.10) have the form

$$\psi_i \equiv \tau H\varphi_{x_i} + \varphi H\zeta S_{x_i} - \varphi^2 c_i = 0$$
$$c_i \equiv 2u_{ip}H^2 + \tau H_p u_{ip} - \tau H u_{ipp} \quad (i = 1, 2, 3)$$

is investigated in a more complex manner.

The three first-order equations

$$S_{x_i}a_j - S_{x_j}a_j + \tau\varphi\,(u_{ip}d_j - d_i u_{jp}) = 0 \quad (i, j = 1, 2, 3; \quad i \neq j) \tag{1.13}$$
$$a_1 \equiv Hu_{1p}\,[\tau^2 (H_p/H)_S - 2\tau^2 (u_{1pp}/u_{1p})_S - 2\,\tau^2 (H/\tau)_S]$$
$$d_i = F_i d_1 + F_i'\,(u_{1p}(2H^2 + \tau H_p) - 2\tau H u_{1pp}) - \tau H u_{1p} F_i'', \quad a_i = F_i a_1 \ (i = 2, 3)$$

are obtained from the equations $D_{x_i}\psi_j - D_{x_j}\psi_i = 0 \ (i \neq j)$.

It follows from the fact that the reduction of double-waves to invariant solutions is forbidden that it is necessary that $a_i = \Delta_i a_1 \ (i = 2, 3)$.

But, then,

$$\tau\Delta\,(u_{ip}d_j - u_{jp}d_i) + (-1)^k H a_1 u_k = 0 \tag{1.14}$$
$$(i, j, k = 1, 2, 3; \ i < j, \ k \neq i, \ k \neq j)$$

follows from the second and third equations of system (1.2) and Eqs.(1.13).

After eliminating $d_i \ (i = 1, 2, 3)$ from the latter equations and Eq.(1.3), we obtain that $\tau H a_1 = 0$. Since $\tau H \neq 0$, it follows from the last equation and Eq.(1.14) that $a_1 = 0$, $d_i = F_i d_1 \ (i = 2, 3)$.

It is subsequently necessary to consider two cases: $(F_2')^2 + (F_3')^2 \neq 0$ and $F_2 = F_3 = 0$. In both cases either the condition $H \neq 0$ or the condition for the functional independence of the pressure and entropy is not satisfied, except in the case of the gases with an equation of state (1.23).

Hence, if $H \neq 0$, there are no stationary non-isentropic non-isobaric solutions of the double-wave type which have a functional arbitrariness and are not reduced to invariant solutions.

2°. Let $H = 0$. Since p and S are functionally independent, it follows from the fact that the reduction of the system of Eqs.(1.2), (1.5), (1.7) to invariant solutions is forbidden that Eqs.(1.8) and (1.12) are also satisfied in the case when $H = 0$.

For the subsequent investigation a transformation is made to the new independent variables p, S and x_3 /6/ (without any loss in generality, it is assumed that the inequality $p_{x_1}S_{x_2} - p_{x_2}S_{x_1} \neq 0$) is satisfied, that is, $x_1 = P\,(p, S, x_3)$, $x_2 = Q\,(p, S, x_3)$.

After this transformation, Eqs.(1.1) are written as:

$$BP_p - AQ_p = 0, \quad u_{1p}BP_S - (\tau + u_{1p}A)\,Q_S = 0 \tag{1.15}$$
$$(\tau + u_{2p}B)\,P_S - u_{2p}AQ_S = 0$$
$$(u_{3p}B - \tau Q_{x_3})\,P_S - (u_{3p}A - \tau P_{x_3})\,Q_S = 0$$
$$(u_{2S} - u_{3S}Q_{x_3})\,P_p - (u_{1S} - u_{3S}P_{x_3})\,Q_p = 0$$
$$A \equiv u_1 - u_3 P_{x_3}, \quad B \equiv u_2 - u_3 Q_{x_3}$$

and, moreover,

$$P_p Q_S - P_S Q_p \neq 0 \tag{1.16}$$

By virtue of the inequality (1.16), the solution of the system of Eqs.(1.15) reduces to the integration of a system of two linear equations for a single unknown functions $Q\,(p, S, x_3)$

$$\omega Q_p - \beta\,(u_2 - u_3 Q_{x_3}) = 0 \tag{1.17}$$
$$\omega_S Q_p - \beta\,(u_{2S} - u_{3S}Q_{x_3}) = 0$$

Here,

$$P = -F_2 Q - x_3 F_3 + \chi \tag{1.18}$$
$$\omega \equiv u_1 + u_2 F_2 + u_3 F_3, \quad \beta \equiv \chi' - F_2'Q - x_3 F_3'$$

where $\chi = \chi\,(p)$ is an arbitrary function. From Eq.(1.18), it follows that

$$x_1 + x_2 F_2 + x_3 F_3 = \chi \tag{1.19}$$

while it follows from the commutation condition (1.16), after (1.18) has been substituted into it, that $\beta Q_S \neq 0$. Apart from this, since $\tau = -u_\alpha u_{\alpha p} = -\omega u_{1p}$, then $\omega \neq 0$. Then, after eliminating Q_p from (1.17), one obtains

$$(u_3/\omega)_S Q_{x_3} - (u_2/\omega)_S = 0 \tag{1.20}$$

If $(u_3/\omega)_S \neq 0$, then, by a rotation of the coordinate system, the solution is reduced to an invariant solution (here, the intermediate calculations are quite unwieldy and have been omitted). In this connection, it is necessary to assume that

$$(u_i/\omega)_S = 0 \quad (i = 2, 3) \tag{1.21}$$

Since double-waves are being considered which are not reduced to planar flows /9/, the relationships

$$u_i = h_i(p)\psi(S) \quad (i = 1, 2, 3) \tag{1.22}$$

and

$$\tau = -(h_\alpha h_\alpha')\psi^2, \quad H = -(h_\alpha h_\alpha'')\psi^2 = 0$$

follow from Eqs.(1.9), (1.12) and (1.21).

Hence, Eqs.(1.22) are only valid in the case of equations of state of the form

$$\tau = g(p)\psi^2(S) \tag{1.23}$$

Therefore, if $H = 0$, it is necessary that Eqs.(1.22) and (1.23) and the relationships

$$h_\alpha h_\alpha'' = 0, \quad h_\alpha h_\alpha' = -g \tag{1.24}$$

$$-gQ_p + h_1'(\chi' - Q(h_2'/h_1')' - x_3(h_3'/h_1')')(h_3Q_{x_3} - h_2) = 0 \tag{1.25}$$

should be satisfied for the existence of stationary non-isentropic non-isobaric gas flows of the double-wave type which have a functional arbitrariness and are not reduced to invariant solutions.

A function Q with an arbitrariness in one function of the two arguments (the argument here is the entropy S) is found from Eqs.(1.25). Since the velocity of sound $c^2 = -g^2\psi^2/g'$ in the case of a gas with an equation of state (1.23), then, by virtue of Eqs.(1.24) and the Cauchy inequality, $c \leqslant |\mathbf{u}|$. Consequently, the flows which are described by these solutions are always supersonic.

The resulting double-waves may be considered as a generalization of simple waves to non-isentropic flows, and, hence, it is possible with their help to construct flows around profiles which are more complex than developing profiles /6, 10/. Solutions of the form of (1.22) have been obtained in /7/ in the case of the equation of state of a polytropic gas $(g = p^{-1/\gamma})$ In the planar stationary case $(h_3 = 0)$ these solutions describe generalized Prandtl-Meyer motions /11/.

2. ***Planar transient double-waves.*** Planar non-steady-state isentropic flows of a polytropic gas with a functional arbitrariness have been investigated in /8/. The following theorem holds in the case of non-isentropic double-waves with an arbitrary equation of state /12/.

Theorem 1. Planar non-isentropic non-isobaric gas flows of the double-wave type which are not reduced to invariant solutions and have a functional arbitrariness can only be of the following forms:

1) double-waves with a single functional arbitrariness in a single argument with an equation of state $\tau = A_1(S)g(p) + A_2(S)$ and, at the same time, $u_i = u_i(S)$ $(i = 1, 2)$ are arbitrary functions, the pressure is determined from the equation $g(p) = c_1t + c_2$ (where c_i are constants and $c_1 \neq 0$) while the entropy satisfies the system of differential equations

$$dS/dt = 0, \; u_\alpha' S_{x_\alpha} = c_1A_1/((c_1t + c_2)A_1 + A_2)$$

which is found in the involution;

2) double-waves with a two-function arbitrariness in a single argument in which the functions $\tau = \tau(p, S), u_i = u_i(p, S)$ $(i = 1,2)$ satisfy the equations (the summation is carried out from 1 to 2)

$$\begin{gathered}\tau u_{\alpha S}u_{\alpha pS} + (\tau_S - \zeta)\zeta = 0 \\ \tau(u_{2S}u_{1pS} - u_{1S}u_{2pS}) + u_{\alpha p}u_{\alpha S}(u_{1p}u_{2S} - u_{2p}u_{1S}) = 0 \\ \tau_p + u_{\alpha p}u_{\alpha p} = 0 \\ d/dt = \partial/\partial t + u_\alpha\partial/\partial x_\alpha, \quad \zeta = \tau_S + u_{\alpha p}u_{\alpha S}, \quad \xi = \tau_S + 2u_{\alpha p}u_{\alpha S}\end{gathered} \tag{2.1}$$

If Eqs.(2.1) are satisfied then, after transforming to the independent variables p, S and t $(x_i = P_i(p, S, t)$ $(i = 1, 2))$, a solution is obtained with the straight contour lines

$$\begin{gathered}u_{\alpha p}u_{\alpha S}P_i = t(u_{iS}(\tau + u_\alpha u_{\alpha p}) + (-1)^i u_{jp}(u_{1S}u_2 - u_{2S}u_1)) + \\ u_{iS}\chi + (-1)^i u_{jp}\Phi \quad (i, j = 1, 2; \; i \neq j) \\ \chi = \Delta^{-1}(\tau\Phi_p - \Phi(u_{\alpha p}u_{\alpha S}\zeta - \Delta^2)/(u_{\alpha S}u_{\alpha p})) \\ \Delta \equiv u_{1S}u_{2p} - u_{2S}u_{1p}\end{gathered}$$

The function $\Phi = \Phi(p, S)$ is found from the second-order equation which is obtained after substituting χ into the equation

$$\tau\chi_S + \Phi\xi\Delta/(u_{\alpha S}u_{\alpha S}) - \zeta\chi = 0$$

The system of Eqs.(2.1) for the functions $\tau(p, S)$ and $u_i(p, S)$ $(i = 1, 2)$ is found in the involution and has an arbitrariness in five functions of a single argument. If, however, the question is posed as to with which specified equations of state $\tau = \tau(p, S)$ this system in $u_i(p, S)$ $(i = 1, 2)$ is compatible and what will be the solution in this case, then a complete investigation is rather difficult. However, in the special case of the equation of state (1.23), the above question can be completely answered. On account of the large number of intermediate calculations we shall only present the route for obtaining the answer to this question.

Initially, it is proved that, if the equation of state has the form of (1.23), then it follows from (2.1) that $u_{2p} = F(p)u_{1p}$. After, this, relationships of the form of (1.22) $u_i = h_i(p)\psi(S)$ $(i = 1, 2)$ follow from the fact that reduction is forbidden, the latter equation and Eqs.(2.1) and the functions $h_i(p)$ $(i = 1, 2)$ satisfy the system of ordinary differential equations

$$g + h_\alpha h_\alpha' = 0, \quad g' + h_\alpha' h_\alpha' = 0$$

However, when the latter equations are satisfied, the solution is reduced to the stationary case. Hence, in the case of equations of state (1.23), solutions are obtained which are reduced to planar solutions and generalized Prandtl-Meyer waves. These solutions have been studied in detail in /11/.

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